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OPTIMIZATION OF INTERDEPENDENT QUEUEING MODEL WITH BULK SERVICE HAVING VARIABLE CAPACITY

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Abstract

In this paper it deals with an interdependent queueing models with bulk service having variable capacity. In this paper two types of interdependent queueing models with bulk service are studied namely Queueing models with arrival and service process being poissonian and follows a bivariate Poisson process and second is Queueing models with arrival process in poissonian and service process is Earlangian having interdependence on each other . The system characteristics of these models are analyzed.

Keywords: The system characteristics like, mean queue length, variability of the system size, coefficient of variation are derived and analyzed in the light of the dependence parameter. These models also include the earlier models as particular cases for specific values of the parameters.

1. Introduction

In the bulk service queueing models, bailley (1954) and Jaiswal (1960) considered units arrive at random from a single queue in order of arrival and are served in batches, the size of each batch being either a fixed number of customers or the whole queue length whichever is smaller. Jaiswal (1961) extended this model to the case, where at a service epoch if m $(0\delta m\delta s)$ persons or the whole queue length whichever is smaller will be taken into service. However, in these models the arrivals and service processes are independent. But in some situations, like an elevator or at a bus stop, etc. the service process depends on the arrival process in order to have optimal operating policies. So, for this kind of situations, we develop and analyze the interdependent queueing models with service. In this paper, two modifications are considered namely (i) the arrival and service processes are poissonian and follows a Poisson processes and (ii) the arrival and services processes are poissonian and Erlangian respectively with interdependent arrival and service processes. In both the models the system behavior is analyzed by obtaining the difference – differential equations of the model and solving them through generating function techniques.

2. $M/M^{[X]}/1$ Interdependent Queuing model with Variable Service Capacity

In this paper, we consider the single server queueing system with interdependent arrival and service process having the bulk service with variable capacity. Here, we assume that the server serves only at instants $t_1, t_2 \dots\dots\dots, t_n \dots\dots\dots$ if $m\ (0\ \delta\ m\ \delta\ s\ )$ persons are present in the waiting line at time t_n then the server takes a batch of $(s-m)$ persons or whole queue length whichever is smaller, where s is the service capacity. Let b_m be the probability that there are m customers remaining with the server at a service epoch($\sum b_m = 1$). Here, we assume that the arrival of the customers and the number of service completions of the batches are correlated and follows a bivariate Poisson process. Let $P_n(t)$ be the probability that there are n customers in the queue at at time t. With dependence structure, the difference – differential equations of the model are

$$\mathrm{P}_0^{'}(\mathrm{t}) = -(\lambda + \mu - 2 \in)\mathrm{P}_0(\mathrm{t}) + (\mu - \epsilon) \sum_{\mathrm{m}=0}^{\mathrm{s}} \mathrm{B}_{\mathrm{s-m}} \mathrm{P}_{\mathrm{m}}(\mathrm{t})$$

$$P_n^{'}(t) = -(\lambda + \mu - 2 \in)P_n(t) + (\lambda - \epsilon)\ P_{n-1}(t) + (\mu - \epsilon)\sum_{m=0}^{s} B_{s-m}\, P_{n+m}(t) \qquad \text{, For n >0} \tag{2.1}$$

Where $B_m\ \sum_{q=0}^{m} B_q$ (is the probability that there are m or fewer customers present with the server). Assuming the steady-state of the system, the steady-state transition equations of the model are.

$$(\lambda + \mu - 2 \in)P_0(t) + (\mu - \epsilon)\ \sum_{m=0}^{s} B_{s-m}\, P_m(t) = 0$$

$$(\lambda + \mu - 2 \in)P_n(t) + (\lambda - \epsilon)\ P_{n-1}(t) - (\mu - \epsilon)\ \sum_{m=0}^{s} B_{s-m}\, P_{n+m}(t) = 0 \tag{2.2}$$

We solve these steady-state equations using generating function approach,

Let $(y) = \sum_{n=0}^{\infty} y^n P_n$, be the generating function of P_n .

Following the heuristic argument of Jaiswal (1961), we get the probability generating function of P_n as

$$P(y) = \frac{(\mu-\epsilon)\left[\sum_{q=0}^{s-1}\{y^s \emptyset_{s-q}(1) - y^q \emptyset_{s-q}(y)\}P_q\right]}{y^s[(\lambda+\mu-2\in)-(\lambda-\epsilon)\ y] - (\mu-\epsilon)\ \sum_{m=0}^{s} b_m y^m} \tag{2.3}$$

Using Roche's theorem, the denominator of the equation (2.3) can be shown to have (s-1) zeros inside the unit circle, one at $y = 1$and the remainder outside the unit circle $[y] = 1$. However, this requires the condition

$$(\lambda - \epsilon)\ < (s - \sum_{m=0}^{s} m\, b_m)(\mu - \epsilon) \tag{2.4}$$

This condition is obviously statistical equilibrium
therefore $P(y)$ can be written as

$$P(y) = \frac{C}{\prod(y - y_i)} \tag{2.5}$$

Where y_i the roots of modulus are greater than one and the product should carry out over all roots with modulus greater than one of the equation

$$y^s - \sum_{m=0}^{s} b_m y^m \left\{\frac{\mu-\epsilon}{(\lambda+\mu-2\epsilon)-(\lambda-\epsilon)\ y}\right\} = 0 \tag{2.6}$$

Using the boundary condition, p (1) =1, we obtain $C = \prod(1 - y_i)$ (2.7)

Where y_i is as given in equation (2.6) Thus, we have

$$P(y) = \frac{(\lambda+\mu-2\epsilon)-(\lambda-\epsilon)\ y-(\mu-\epsilon)}{(\lambda-\epsilon)(1-y)} \prod\left(\frac{1-y_i}{y-y_i}\right) \tag{2.8}$$

Where y_i is as given in equation (2.6)

Using the probability generating function, we can analyze the system behavior of this model. Expanding equation (2.8) and collecting the coefficient of y^{n}, will give us the probability that there are n customers in the system.

3. Measures of Effectiveness

The probability that the system is empty can be obtained as

$$P_0 = \prod\left(1 - \frac{1}{y_i}\right) \tag{3.1}$$

Where y_i's are as given in equation (2.6)

For given values of λ and μ and for various values of ϵ and s , and for given values of s and ϵ and for various values λ and μ the values of P_0 are computed and are given in tables (01) and (02)

Table (01) Values of P_0 λ =1, μ =6

s/ϵ	0	0.2	0.4	0.6	0.8
1	0.6667	0.7241	0.7857	0.8333	0.9867
2	0.7532	0.8110	0.8505	0.8945	0.9439
3	0.8055	0.8536	0.8693	0.9072	0.9504
4	0.8194	0.8471	0.8782	0.9134	0.9536
5	0.8273	0.8537	0.8833	0.9170	0.9555

Table (02) Values of P_0 S=3, ϵ = 0.4

μ/λ	1	2	3	4
5	0.8437	0.6389	0.4767	0.3412
6	0.8693	0.6920	0.5479	0.4259
7	0.8877	0.7313	0.6016	0.4903
8	0.9015	0.7617	0.6436	0.5411

From table (01) and equation (2.9), it is observed that the value of P_0 increases as ϵ increases for fixed values of λ , μ and. It is also noticed from table (01) and equation (2.9) that the p increases as the batch size s increases for fixed values of λ , μ and ϵ .The values increases of P_0 increases as the service rate μ increases for given values of λ , ϵ and s . It decreases as the arrival rate increases for fixed values of s , μ and ϵ .

The average number of customers in the system can be obtained by differentiating P(y) with respect to y and substituting y=1.From equation (2.8) and L-Hospital's rule, we have

$$L = \sum \frac{1}{1-y_i} \tag{3.2}$$

Where y_i's are as given in equation (2.6)

Using equation (10), we have computed the values of L for various values of ϵ and s and for fixed values of λ and μ are presented in table (03). For fixed values of s, ϵ and for various values of λ and μ and the values of L are given in table (04). From table (03) and eqn. (3.2) it is observed that the average number of

Table (03) Values of λ =1, μ =6

$s/\in$	0	0.2	0.4	0.6	0.8
1	0.5000	0.3810	0.2727	0.2000	0.0135
2	0.3278	0.2331	0.1758	0.1180	0.0595
3	0.2415	0.1968	0.1504	0.1022	0.0522
4	0.2205	0.1806	0.1387	0.0948	0.0487
5	0.2087	0.1714	0.1321	0.0905	0.0466

Table (04) Values of L S=3, $\in$ = 0.4

μ/λ	1	2	3	4
5	0.1853	0.5653	1.097	1.9305
6	0.1504	0.4450	0.8251	1.3477
7	0.1265	0.3674	0.6623	1.0395
8	0.1092	0.3128	0.5537	0.8482

Customers decreases as the dependence parameter increases, for fixed values of, μ and s . It is also noticed that the average number of customers in the system decreases as the as the batch size s increases for fixed values of λ , μ and ϵ . From table (03) and equation (3.2) it is observed that as increases the average number of customers in the system increases for fixed values of s , μ and ϵ . As the service rate increases the average number of customers in the system decreases For fixed values of s , λ and ϵ**.** The variability of the system size is obtained by the formula

$$V = p''(z) + p'(z) - [p'(z)]^2 \Big|_{z=1} \tag{3.3}$$

Differentiating P (z) and putting z=1, we get,

$$V = \sum \frac{1}{(y_i-1)^2} + \sum \frac{1}{y_i-1} \tag{3.4}$$

Where y_i's are as given in equation (2.6)

$$C.V = \sqrt{\sum \frac{y_i}{(y_i-1)^2}} \Big/ \sum \frac{1}{y_i-1} \tag{3.5}$$

Where y_i's are as given in equation (2.6)

For fixed values of, μ and for various values of ϵ and s the values of the variability of the system size and coefficient of variation are given in table (05) and table (07). The computed values of V and C.V for fixed ϵ and s and for varying λ and μ are given in tables (06) and (08).

From tables (2), (3) and from equations (3.4) and(3.5) it is observed that the variability of the system size decreases as the dependence parameter increases and coefficient of variation increases for fixed values of s , μ and λ . As the batch size s increases, the variability decreases and coefficient of variation increases. From tables (06) and (08) and from equations (3.4) and(3.5) it is observed as the arrival rate increases the variability increases and coefficient of variation decreases for fixed values of μ, ϵ and s .
This model includes the earlier models as particular cases for the specific values of the parameters. This model becomes $M/M^{[X]}/1$ model with bulk service rule when $\in = 0$.

Table (05) Values of V λ =1, μ =6

$s/\in$	0	0.2	0.4	0.6	0.8
1	0.7500	0.5261	0.3471	0.2400	0.2137
2	0.4352	0.2874	0.267	0.1319	0.0630
3	0.2999	0.2355	0.1730	0.1127	0.0549
4	0.2691	0.2132	0.1580	0.1038	0.0510
5	0.2523	0.2008	0.1495	0.0987	0.0488

Table (06) Values of S=3, $\in$ = 0.4

μ/λ	1	2	3	4
5	0.2197	0.8848	2.3026	5.6763
6	0.1730	0.6431	1.5058	3.1640
7	0.1425	0.5023	1.1008	2.1201
8	0.1212	0.4108	0.8603	1.5678

Table (07) Values of C.V λ =1, μ =6

$^{S}/_{\in}$	0	0.2	0.4	0.6	0.8
1	1.7321	1.9039	2.1603	2.4495	3.6603
2	2.0127	2.3000	2.5860	3.0781	4.2213
3	2.2672	2.4663	2.7658	3.2833	4.4901
4	2.3528	2.5570	2.8649	3.3976	4.6419
5	2.4065	2.6143	2.9279	3.4708	4.7397

Table (08) Values of C.V S=3, $\in$ = 0.4

$^{\mu}/_{\lambda}$	1	2	3	4
5	2.5290	1.6641	1.3823	1.2321
6	2.7659	1.8019	1.4873	1.3198
7	2.9838	1.9292	1.5843	1.4007
8	3.1865	2.0484	1.6741	1.4761

M/M /1 Model when $\in$ = 0, $= 1$, $b_0 = 1$, $b_1 = 0$.

M/M /1 Interdependence model $s = 1$, $b_0 = 1$, $b_1 = 0$.

4. $M/{E_K}^{[X]}/1$ Interdependent Model

We consider the single server queueing system where the service is in phases. Along with other assumptions we assume that the number of arrivals and the number of service completions in each phase are correlated and follows a bivariate poission distribution. As in the earlier, here we assume the service is available at time instants $t_1 , t_2 \dots \dots \dots , t_n$.

Let b_m be the probability that there are m customers present with the server in the system at a service epoch. Then the server takes (s-m) customers or the whole queue length whichever is smaller. S is the maximum size of the batch that is to be taken into service.

We have $b = 0$ if $m > s$ and $\sum_{m=0}^{s} b_m = 1$. With this dependence structure we develop

$M/{E_K}^{[X]}/1$ Interdependent model with bulk service.

Let $P_n(t)$ be the probability that there are n customers waiting in the queue at time t and service in the r^{th} phase. Using the phase - type technique, we can have the differential equation of the model as

$P_{n,i}{}'(t) = -(\lambda + \mu - 2 \in)P_{n,i}(t) + (\mu - \epsilon)P_{n,i+1}(t) + (\lambda - \epsilon)P_{n-1,i}(t) + (\mu - \epsilon)\sum_{m=0}^{s} b_{s-m}P_{n+m,1} = 1(t)$
$n > 0$ And $i < k$

$P_{0,i}{}'(t) = -(\lambda + \mu - 2 \in)P_{0,i}(t) + (\mu - \epsilon)P_{0,i+1}(t) + (\mu - \epsilon)\sum_{m=0}^{s} B_{s-m}P_{m,1} = 1(t)1 < r, 1 > k$ (4.1)

Where $b_m = \sum_{q=0}^{s} b_q$

Assuming the system is in steady-state, the state transition equations of the model are

$$-(\lambda+\mu-2\in)P_{n,i}+(\mu-\epsilon)P_{n,i+1}(\lambda-\epsilon)P_{n,i+1}+(\mu-\epsilon)\sum_{m=0}^{s}b_{s-m}P_{n+m,1}=0$$

$$-(\lambda+\mu-2\in)P_{0,i}+(\mu-\epsilon)P_{0,i+1}(\lambda-\epsilon)\sum_{m=0}^{s}B_{s-m}P_{m,1}=0 \qquad (4.2)$$

To solve these steady-state equations, we adopt the generating function approach.

Let $\sum_{n=0}^{\infty} y^n P_n$ (4.3)

be the generating function of P_n .

Following the heuristic argument of Jaiswal (1961), we get the probability the generating function of P_n as,

$$P_n(y)=\frac{(\mu-\epsilon)^k\left[\sum_{q=0}^{s-1}\{y^s\phi_{s-q}(1)-y^q\phi_{s-q}(y)\}P_q\right]}{[(\lambda+\mu-2\in)-(\lambda-\epsilon)\,y]^k y^s-(\mu-\epsilon)^k\sum_{m=0}^{s}b_m y^m} \qquad (4.4)$$

Applying Roche's Theorem for the denominator, we get

$$[(\lambda+\mu-2\in)-(\lambda-\epsilon)\,y]^k y^s-(\mu-\epsilon)^k\sum_{m=0}^{s}b_m y^m \qquad (4.5)$$

It can be shown to have (s-1) zeros inside the unit circle and one at $y=1$and the remainder outside the unit circle $|y|=1$. However this requires the condition,

$$k(\lambda-\epsilon)<(\mu-\epsilon)\left[s-\sum_{m=0}^{s}mb_m\right] \qquad (4.6)$$

The condition is necessary for statistical equilibrium. Thus $P(y)$ can be written as

$$P(y)=\frac{C}{\prod(y-y_i)}$$
(4.7)

Where $y_i's$ are roots of the modulus greater than one of the equation?

$$[(\lambda+\mu-2\in)-(\lambda-\epsilon)\,y]^k y^s-(\mu-\epsilon)^k\sum_{m=0}^{s}b_m y^m=0 \qquad (4.8)$$

Using the boundary condition p (1) =1, we obtain,

$$C=\prod(1-y_i) \qquad (4.9)$$

Thus,

$$P(y)=\frac{[(\lambda+\mu-2\in)-(\lambda-\epsilon)\,y]^k-(\mu-\epsilon)^k}{(\mu-\epsilon)^{k-1}(\lambda-\epsilon)\,(1-y)}\prod\left(\frac{1-y_i}{y-y_i}\right) \qquad (4.10)$$

Where $y_i's$ are given in equation (4.8)
Using the probability generating function, we can analyze the system behavior of this model. Expanding the equation (4.10) and collecting the coefficient of y^n, will give us the probability that there are n customers in the system.

5. Measures of Effectiveness

The probability that the system is empty can be obtained as

$$P(y) = \frac{(\lambda+\mu-2\in)^k-(\mu-\epsilon)^k}{k(\mu-\epsilon)^{k-1}(\lambda-\epsilon)} \prod\left(1-\frac{1}{y_i}\right) \tag{5.1}$$

Where $y_i{}'s$ are given in equation (4.8)
Using the equation (5.1), the values of P_0 are computed for various values of s and for fixed values of λ , μ and k and are given in table (09). From table (09) it is observed that the values of P increases as ϵ increases for fixed values of λ , μ and k. It is also noticed that the values of P_0 increases as the batch size s increases for fixed values of λ , μ and k

Table (09) Values of P_0 K=2, λ =1, μ =6

$s/\in$	0	0.2	0.4	0.6	0.8
1	0.3611	0.4792	0.7716	0.7294	0.8781
2	0.6270	0.6868	0.7533	0.8252	0.9069
3	0.6938	0.7403	0.7925	0.8519	0.9202
4	0.7228	0.7640	0.8107	0.8463	0.9266

The average number of customers in the system can be obtained by differentiating P(y) with respect to y and substituting y=1.

From equation (2.2, 2.4 and 3.3) and L-Hospital's rule, we have

$$L = \sum\left(\frac{1}{y_i-1}\right) + \frac{(k-1)}{2}\left[\frac{\lambda-\epsilon}{\mu-\epsilon}\right] \tag{5.2}$$

Where y_i's are as given in equation (4.8)

Using equation (5.2) we have computed the values of L for various values of ϵ and s and are presented in table (10).

From table (10) it is observed that the values of L decrease as ϵ increases for fixed values of λ , μ and k. And it is also noticed from table (10) that the values of L decreases as the batch size increases.

Table (10) Values of L K=2, λ =1, μ =6

$s/\in$	0	0.2	0.4	0.6	0.8
1	1.7499	1.0767	0.2923	0.3689	0.1408
2	0.5857	0.4502	0.3259	0.2104	0.1023
3	0.4335	0.3457	0.2588	0.0.1725	0.0864
4	0.3762	0.3030	0.2306	0.1556	0.0789

The variability of the system size is obtained by using the formula (11). Thus we have,

$$V = \left[\sum \frac{1}{(y_i-1)^2} + \sum \frac{1}{y_i-1} + \left(\sum \left[\frac{1}{y_i-1}\right]\right)^2\right] + \frac{1}{3}(k-1)(k-2)\left(\frac{\lambda-\epsilon}{\mu-\epsilon}\right) - (k-1)\left(\frac{\lambda-\epsilon}{\mu-\epsilon}\right)\sum \frac{1}{y_i-1} + \frac{(k-1)}{2}\left[\frac{\lambda-\epsilon}{\mu-\epsilon}\right] - \left[\sum \frac{1}{y_i-1} + \frac{(k-1)}{2}\left[\frac{\lambda-\epsilon}{\mu-\epsilon}\right]\right]^2 \quad (5.3)$$

Using equation (5.3) we have computed the values of V for various values of k and ϵ and for for fixed values of λ, μ and ϵ and also for various values of s and ϵ and for fixed values of λ, μ and k.

The coefficient of variation of the system is

$$C.V = \sqrt{V}/L$$

Where V and L are as given in equations (5.2) and (5.3) respectively. The values of V and C.V are computed for various values of, s and for fixed values of of λ, μ and k and are presented in table (11) and (12).

Table (11) Values of V K=2, λ =1, μ =6

$s/\in$	0	0.2	0.6	0.8
1	4.7704	2.2149	0.5008	0.1948
2	0.9091	0.6405	0.2515	0.1120
3	0.6047	0.4542	0.1993	0.0931
4	0.5019	0.3860	0.1770	0.0843

Table (12) Values of K=2, λ =1, μ =6

$s/\in$	0	0.2	0.6	0.8
1	1.2481	1.3822	1.9183	2.8825
2	1.6279	1.7775	2.3839	3.2709
3	1.7939	1.9494	2.5881	3.5220
4	1.8834	2.034	2.7031	3.6823

It is observed that as s increases the variability of the system size decreases and the coefficient of variation increases as the dependence parameter increases the variability of the system size decreases and the coefficient of variation increases for fixed values of λ, μ and k. It is also noticed that as k increases the system variability increases for fixed values of λ, μ, ϵ and s and the coefficient of variation decreases for fixed values of λ, μ, ϵ and s. As the dependence parameter increases the variability of the system size decreases and the coefficient of variation increases for fixed values of λ , μ ,k and s.

6. Conclusions

This paper considered the single server queueing system with interdependent arrival and service process having the bulk service with variable capacity. The arrival of the customers and the number of service completions of the batches are correlated and follows a bivariate Poisson process. The average number of customers in the system decreases as the as the batch size. Increases.. As the service rate increases the average number of customers in the system decreases It is observed that the variability of the system size decreases as the dependence parameter increases and coefficient of variation increases .As the batch size s increases, the variability decreases and coefficient of variation increases. This model includes the earlier models as particular cases for the specific values of the parameters.

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